Some numerical methods for resolution of non-linear integral equations*

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ABSTRACT. We suggest several methods which bring back the resolution of non-linear integral equations to a succession of linear integral equations or to a linear algebraic system. All these methods have been tested numerically.

1. Introduction

We will examine integral equations such that

\[ u(x) = \lambda \int_a^b K(x, t) \cdot g[u(t)] \, dt + f(x) \]  

where, functions \( K, g, f \) are given.

We must calculate \( u(x) \). We are concerned by the search of simple numerical methods for the determination of \( u(x) \).

We will suppose that Eqn. (1) admits a unique real solution for \( f, K, g \), given. Theorems of existence and unicity are given by Miklin and Tricomi. In Anselone, Cherruault and Glowinski's publications, we find numerical methods which bring through the resolution of non-linear algebraic systems. In the best case, R. Glowinski obtains a succession of non-linear algebraic equations. We will give here simpler methods, easy to use.

2. First method

Reduction to a succession of linear integral equations (Cherruault et Guilleux):

Starting from Eqn. (1) we build \( u_0(x) \) solution of

\[ u_0(x) = \lambda \int_a^b K_0(x, t) \cdot g[u_0(t)] \, dt + f(x) \]  

where, \( K_0(x, t) \) is an approximation of \( K(x, t) \) which may be crude.

Let us set \( R(x, t) = K(x, t) - K_0(x, t) \) and

\[ n(x) = \lambda \int_a^b R(x, t) \cdot g[u_0(t)] \, dt. \]

It is easy to show that \( v(x) = u(x) - u_a(x) \) is solution of

\[ v(x) = n(x) + \lambda \int_a^b K(x, t) \cdot [g(u) - g(u_a)] \, dt \]  

We can notice that \( n(x) \) is known if \( u_0 \) is calculated.

Setting \( g(u) - g(u_a) = (u - u_a) \cdot G(u, u_a) \cdot v(t) \) we obtain the integral equation

\[ v(x) = n(x) + \lambda \int_a^b K(x, t) \cdot G(u, u_a) \cdot v(t) \, dt \]  

The first method will give the result of an interesting approximation of \( G(u, u_a) \).

Define the numerical algorithm as below:

Setting \( u_{n+1} = u_0 + v_n \)

\[ u_1(x) = u_0 + v_n \]  

is obtained from

\[ v_n(x) = n(x) + \lambda \int_a^b K(x, t) \cdot G(u, u_a) \cdot v_n(t) \, dt \]

which is a linear integral equation.

Generally \( v_n(x) \) is given by resolution of the linear integral equation:

\[ v_n(x) = n(x) + \lambda \int_a^b K(x, t) \cdot G(u, u_a) \cdot v_n(t) \, dt \]  

There exist a lot of simple numerical methods to solve the Eqn. (5).

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Practically, as we will see later, this method converges quickly (in three of four iterations). Convergence may be improved by using "good" approximations for $G(u_0, u)$.

The following can be established (Cherraualt et Guillez)

**Theorem 2.1**—If we suppose $| G(u_0, u) | \leq G < \infty$, $K(x, t) \leq \kappa - \infty$ and $n(x) \in H^1(a, b)$, then a solution of Eqn. (5) converges towards $u(x)$ solution of Eqn. (1).

For the proof we use bounded sequences and weak compactness.

It is possible to apply this method to more general integral equations.

For instance we can deal with:

(a) $u(x) = f(x) + \int_a^b K(x, t) \cdot g(u, t) dt$

(b) $u(x) = f(x) + \lambda \int_a^b K[x, t, u(t)] dt$

3. A second method

This method leads to the resolution of a sequence of linear systems. It can be considered as a discretization of the previous method.

Let us consider the equation

$$u(x) + \int_0^1 K(x, t) \cdot u^3(t) dt = f(x)$$  \hspace{1cm} (6)

We can, of course, approximate Eqn. (6) by:

$$u(x) + \sum_{j=0}^{N-1} K(ih, t) \cdot u^3(t) dt = f(x)$$  \hspace{1cm} (7)

which gives for $x = ih$

$$u(ih) + \sum_{j=0}^{N-1} K(ih, t) \cdot u^3(t) dt = f(ih)$$  \hspace{1cm} (8)

On $[jh, (j+1)h]$ let us approach $u^3(t)$ by

$$u^3(ih) + 3 u^2(ih) \cdot [u((j+1)h) - u(ih)]$$

It's a linearization near $t = ih$. It's valid because $u^3(t)$ is used on an interval of length $h$ with $h \to 0$.

**Remark 1**—If we had $g(u)$ instead of $u^3(t)$, we could approximate $g(u)$ by $g(u_0) + (u - u_0) \cdot g'(u_0)$.

Let us now build the iterative diagram.

$$u_n(ih) + \sum_{j=0}^{N-1} K(ih, (j+1)h) \cdot \{ u^3_{n-1}(j) +$$

$$+ \frac{3}{2} u^2_{n-1}(j) \cdot [u_n((j+1)h) - u_{n-1}(j)] \} = f(ih)$$

$$i=0, \ldots, N; j=0, \ldots, N-1, n=1, 2, \ldots$$  \hspace{1cm} (9)

where $u_n(jh)$ have been calculated before or are arbitrarily fixed.

For a fixed $n$ Eqn. (9) is a linear system whose $u_n(jh)$ are unknown ($j = 0, \ldots, N$). The system (9) will give an approximate solution or (6) or (7) by successive resolution of linear systems.

For convergence proof, we can use the same method as in §2. Nevertheless it will be necessary to take into account the discretization.

**Remark 2**—The scheme (9) is not unique. More generally, we can choose more sophisticated approximations of

$$\int_{jh}^{(j+1)h} \quad \text{and of } u^3(t) \text{ or } g(u).$$

**Remark 3**—Practically it will be useful to start with well adapted $u_0(jh)$. A. Caron (Lab. Medimat, see Ref.) has noticed that a good approximation was given by:

$$u_0(ih) = \frac{f(ih)}{1 + (e^{(j+1)h} - 1|i|)h}$$  \hspace{1cm} (10)

$$u_0(0) = \frac{f(0)}{2}$$

with the kernel $K(x, t) = e^t$. In general case we use

$$u(x) = \frac{f(x)}{1 + \int_{0}^{x} K(x, t) dt}$$  \hspace{1cm} (11)

Moreover it's possible to suggest a method involving resolution of a sequence of linear equations instead of the previous linear system. In fact let's associate to Eqn. (8) the iterative scheme:

$$u_n(ih) + \sum_{j=0}^{i-1} \int_{jh}^{(j+1)h} K(ih, t) \cdot u_n^3(t) dt +$$

$$+ \sum_{j=0}^{N-1} \int_{jh}^{(j+1)h} K(ih, t) \cdot u_{n-1}^3(t) dt = f(ih)$$

$$i=1, \ldots, N-1$$  \hspace{1cm} (12)
Then, we have
\[ \int_{j}^{(j+1)h} u_a^3(t) \cdot K(ih, t) \, dt \]
\[
\equiv h \cdot K(ih, (j+1)h) \cdot [u_{n-1}^3(jh)+3u_{n-1}^2(jh) \cdot u_{n-1}(jh)] +
[u_{n-1}((j+1)h)-u_{n-1}(jh)]
\]
so that (12) becomes
\[ u_n(ih)+\sum_{j=0}^{L} h \cdot K[ih, (j+1)h] \cdot [u_a^3(jh)+3u_a^2(jh) \cdot u_a(jh)] +
\sum_{j=0}^{N-1} \int_{j}^{(j+1)h} K(ih, t) \cdot u_a^3(t) \, dt = f(ih) \]
\[ i=1, \ldots, N. \tag{13} \]

where \( u_0 \) are fixed or calculated as in remark 3.

Eeqn. (13) shows us that we obtain \( u_n(h) \), \( u_4(2h) \), \ldots by resolution of a sequence of linear equations.

**Remark 4**—As for previous method it is possible to improve the approximation of
\[ \int_{j}^{(j+1)h} K(ih, t) \cdot u_a^3(t) \, dt. \]

Moreover Eqn. (13) is not complete without \( u_a(0) \). We have taken
\[ u_n(x) = -\sum_{j=0}^{N-1} \int_{j}^{(j+1)h} K(ih, t) \cdot u_a^3(t) \, dt + f(0). \]

4. Numerical results*

For the method described in § 2, numerical trials have been done on a pocket calculator.

The following equations have been solved
\[ u(x) = x + \int_{-1}^{1} e^{xt} \cdot u_a^3(t) \, dt \]

We have chosen \( K_0 = 1 + xt, \ R = e^{xt} - 1 - xt \)
\[ u(x) = f(x) - \int_{-1}^{1} e^{xt} \cdot u_a^3(t) \, dt \]

with \( f(x) = \frac{e-x-1}{x} + 1 \) which implies \( u(x) = 1 \)

We have taken \( K_0 = 1 \) which is a crude approximation of \( e^{xt} \). Errors (in \( L^2 \) norm) are between 1.10^{-2} and 1.10^{-4}.

With the methods described in § 3 we have solved (see Cherruault et Guillez and Caron):
\[ u(x) = f(x) - \int_{0}^{1} e^{xt} \cdot u_a^3(t) \, dt \]

with
\[ f(x) = 1 + \frac{e-x-1}{x} \cdot [u(x) = 1] \]

then with
\[ f(x) = x + \frac{e-x}{x} - \frac{3e-x}{x^2} + \frac{6e-x}{x^3} - \frac{6(e-x-1)}{x^4} \]

and
\[ f(x) = x^2 + \frac{e-x}{x} - \frac{6e-x}{x^2} + \frac{30e-x}{x^3} - \frac{120e-x}{x^4} + \frac{360e-x}{x^5} \]

\[ -\frac{720e-x}{x^6} + \frac{720(e-x-1)}{x^7} \cdot [u(x) = x^2] \]

Precision is similar to the previous method. It is possible to improve precision by using best quadrature formula (Gauss).

5. Conclusions

The described methods are simple and easy to use because they lead to linear equations or systems. Moreover they can be extended, without difficulties, to integral equations or systems of several variables. We can also apply these methods to singular, non-linear integral equations.

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*The methods of § 3 have been numerically tested by A. Caron (Medimat), (see Ref.)*


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