An analytical model for mountain wave in stratified atmosphere

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ABSTRACT. An analytical, two-dimensional computer model has been developed for real time prediction of mountain wave due to Pirpanjal mountains over Kashmir valley. Simulation of the $L^2$ profile has been made with realistic, non-zero values at higher levels and exponentially decreasing values at lower levels. Unlike Doos (1961), present solution has no restriction on the value of wave number ($k$). Validity of the model has been tested with the satellite observed waves in seven cases and actual aircraft report in one case.

Key words—Atmospheric, Mountain, Wavenumber, Waves.

1. Introduction

The problem of airflow over mountainous regions has been studied intensively for the past several years by using linear and nonlinear models. Lyre (1943) and Queney (1948) probably, provided the starting point for these investigations. They obtained some observed features of airflow over mountains in a simple two-dimensional, single layer model. Scorer (1949) used an atmosphere stratified in two layers. Although this representation of atmosphere is rather simplified, the results were illuminating and gave rise to extensive studies of two or more layer models. Scorer’s model represented constant $L^2$ (Scorer’s parameter) profile in the two layers with discontinuity at the interface. Since then a number of attempts were made to simulate the actual atmospheric profile more realistically.

Although linearized theory of internal gravity waves due to orography is strictly applicable to shallow mountains and small amplitude waves only, it has been widely used to study large variety of mountain shapes and atmospheric profiles, to obtain a fairly good idea of airflow pattern (Doos 1961, Vergeiner 1971, Klemp and Lilly 1975). Philips and Brown (1983) has recently carried out experimental validation of the waves predicted by the linearized theory and actual aircraft observations. They found that linearized theory is good enough in predicting the wave-length. Amplitude of the wave, however, was noticed to be slightly more than what was predicted by the linearized model. Over Indian region Sarkar (1965) studied the waves generated due to Western Ghats.

Over Jammu & Kashmir region, Pirpanjal ranges, having near north-south orientation form a favourable barrier for the generally prevailing westerly winds in this region.

Earlier attempts to simulate the Scorer’s profile as a continuous model may be broadly classified into two categories - an exponentially decreasing single layer model (Foldvik and Palm 1959, Sarkar 1965) and a multilayer exponential model with appropriate matching conditions for vertical velocity and its vertical gradient (Danielson and
Bleck 1970, Palm and Foldvik 1960). Although the later approach is quite close to reality, the relevant expressions become increasingly cumbersome with increasing number of layers. While the former approach is simpler, exponentially decreasing $L^2$-profile not only restricts its applicability to limited number of cases but also does not represent true atmospheric conditions at higher levels. Doos (1961) attempted to study the realistic profile with constant non-zero values (say $L^2_{\infty}$) at higher levels but neglected the contribution to the Fourier integral for values of wavenumbers less than $L^2_{\infty}$. This paper illustrates a method to solve the lee-wave equation with similar stratification but for the entire wavelength spectrum for airflow over the Pirpanjal ranges.

2. Formulation of the problem and solution

Assuming that the mountain is shallow, wave amplitude is small and the airflow is inviscid and laminar, the governing equation for the vertical component of the velocity, under Boussinesq approximation, can be expressed as (Scorer 1949)

$$ \frac{\partial^2 w}{\partial z^2} + \frac{\partial^2 w}{\partial x^2} + L^2 \left( z \right) w = 0 $$

(1)

where

$$ \beta = \frac{1}{\theta} \frac{d\theta}{dz}, \quad S = \frac{1}{\rho} \frac{d\rho}{dz} $$

$$ L^2 \left( z \right) = \left( \frac{g \beta}{U^2} \frac{dU}{dz} + \frac{S dU}{U dz} - \frac{S^2}{4} + \frac{1}{2} \frac{dS}{dz} \right) $$

(2)

Boundary Conditions

(i) Along the mountain ($z = \xi(x)$) the continuity of the normal component of velocity leads to

$$ \frac{W}{U} = \frac{d\xi}{dx} \bigg|_{z=0} $$

(3a)

(ii) For large values of $z$, we will take the more realistic radiation condition implying the upward propagation of energy as $z \to \infty$

$$ w \propto \exp \left( ivz \right), \quad v = \text{a constant} $$

(3b)

To represent the stratification of the atmosphere, we consider a single layer profile given by

$$ L^2 = L^2_{\infty} + L^2_0 \ e^{\alpha z}, \quad \alpha < 0 $$

(4)

Here, $L_{\infty}$, $L_0$ and $\alpha$ are determined by fitting in the observed atmospheric data. Applying Fourier transform to Eqn. (1), we obtain

$$ \tilde{w}_{zz} + \left( L^2 - k^2 \right) \tilde{w} = 0 $$

(5)

where

$$ \tilde{w} \left( k, z \right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} W \left( x, z \right) e^{-ikx} \, dx $$

and

$$ w \left( x, z \right) = \int_{0}^{\infty} \tilde{w} \left( k, z \right) e^{ikx} \, dk $$

Substituting Eqn.(4) in Eqn.(5) we get,

$$ \tilde{w}_{zz} + \left( L^2_{\infty} + L^2_0 \ e^{\alpha z} - k^2 \right) \tilde{w} = 0 $$

(6)

Eqn.(6) can be reduced to the more familiar form

$$ \frac{d^2 \tilde{w}}{d\mu^2} + \frac{1}{\mu} \frac{d \tilde{w}}{d\mu} + \left( 1 - \frac{\nu^2}{\mu^2} \right) \tilde{w} = 0 $$

(7)

where

$$ \mu = \frac{2L_0 \ e^{\alpha z/2}}{\infty} $$

and

$$ \nu = \frac{2(L^2_{\infty} - k^2)^{1/2}}{\infty} = i\rho $$

(8)

The solution of Eqn.(7) satisfying the radiation condition is (cf. Appendix I)

$$ \tilde{w} = AJ_{-\nu} \left( \mu \right) $$

(9)

Where, $A$ is an arbitrary function of the wavenumber $k$.

The mountain profile is taken in the form

$$ \xi(x) = \frac{Ha^2}{a^2 + x^2} + \frac{d}{dx} \left( \frac{bc^2}{c^2 + x^2} \right) + \frac{df^2}{f^2 + (x-36.0)^2} $$

(10)

which represents an analytical expression for the actual ground profile between Peshawar (34.01°N/71.35°E) and Srinagar (34.05°N/74.58°E) across Pirpanjal mountains. Values of the parameters are computed by fitting in Eqn.(10) with the actual topography (Fig.1) by using the method of least square (employing Marquardt numerical technique—Kuestner and Joe 1973) and are given as

$$ H = 1.9; \quad a = 8.6019796; \quad b = 39.353073; \quad c = 49.038303; \quad d = 1.8835262; \quad f = 178.43532; $$

(11)

The mountain profile Eqn.(10) comprises of linear combination of bell-shaped functions (symmetrical with respect to the vertical axis) and its gradient function (asymmetrical with respect to the vertical axis). Addition of two or more of these types of functions would be a convenient analytical tool to formulate complicated ground profile into a simple mathematical model.

Fourier transform of Eqn.(10) gives
\[ \xi (k) = Hae^{-a |k|} + ikbe^{-r|k|} + dfe^{-f |k| + i36k} \]  

and the boundary condition in Eqn. (3) takes the form

\[ \frac{\tilde{w}}{U} = ik \xi (k) \bigg|_{z = 0} \]  

Using Eqn. (13) in Eqn. (9), we obtain

\[ \tilde{w} = \frac{ik \xi (k) U(0)}{J_{ip} (\mu_0)} \]  

Where, \( \mu \bigg|_{z = 0} = \mu_0 \)

Taking inverse Fourier transform of Eqn. (14) we get,

\[ w = \Re \int_{0}^{\infty} \frac{1}{J_{-is} (\mu_0)} \left\{ -ik \left( 1.9ae^{-a|k|} + kbee^{-ik} \right) \right. \]  

+ \left. dfe^{-f |k| + i36.0i} U(0) J_{-is} (\mu) e^{-kx} \right \} dk + 2\pi i (\text{Residue}) \]  

(i) For \( x > 0 \)

\[ w = \Re \int_{0}^{\infty} \frac{1}{J_{-is} (\mu_0)} \left\{ -ik \left( 1.9ae^{-a|k|} + kbee^{-ik} \right) \right. \]  

+ \left. dfe^{-f |k| + i36.0i} U(0) J_{-is} (\mu) e^{kx} \right \} dk \]  

(ii) For \( x < 0 \)

In the above, \( p = s \) along the positive half of the imaginary axis and \( p = s_1 \) along the negative part of the imaginary axis in the complex \( k \)-plane.

\[ \text{Residue} = \sum_{p=1}^{N} \frac{1}{\frac{d}{dk} \left( J_{ip} (\mu_0) \right) k = k_p N} \]

\[ \times \left\{ ik_p N \left( 1.9ae^{-a|k_p N|} + ik_p N bee^{-ek_p N} \right) \right. \]  

+ \left. dfe^{-f |k_p N| + i36.0i} U(0) J_{-ip} (\mu) e^{ik_p N} \right \} \]

According to Coulomb (1936) the zeroes of \( J_{ip} (\mu_0) \) are regarded as real and simple for positive values of the argument \( \mu_0 \). The integral at Eqn. (15) can now be evaluated by using the method of residues.
where \( F_N = P_{k_n k_N} \) and \( K_{PN} \) are the real zeros of \( L_{\alpha p} (\lambda_0) \).

The vertical velocity \( w(x, z) \) is given by Eqn.(16) and Eqn.(17).

### 3. Comparison of the predicted value with observed data

For the purpose of validation of our results we selected AVHRR pictures from NOAA satellite of lee-waves over Kashmir valley, which is situated on the leeward side of Pirpanjal ranges. This range is about 2900 km long (aligned in the 340° - 160° direction) and 50 km wide with average height of 3.6 km. Viewing from the plains of Punjab this is the first mountain range we encounter in the North/Northwest. Peshawar is situated west of this range at about 260 km from the peak and at 0.36 km amsl. East of Pirpanjal is Kashmir valley at 1.5 km amsl. This valley is about 70 km wide. Further east of this valley is the Great Himalayan ranges.

Fig.1 shows the two-dimensional topography along the Peshawar-Srinagar axis up to Srinagar only. With this as the lower boundary, our solution will be valid only up to the Srinagar valley. Actual aircraft reports indicate that turbulence is normally reported at the height of 6-8 km. This level is much above the average height of the other ranges in the region (average height of the Greater Himalayan barrier is 5 km). Effect of the Himalayan barrier, therefore, may be assumed to cause no significant difference on the high level turbulence or wave profiles over Srinagar valley region. Hence, in this study we have neglected the effect of the Greater Himalayan ranges on the lee-waves over the Srinagar valley. Wind and temperature distribution representing the upstream conditions were obtained from the upper air sounding data of Peshawar.

Computer program was developed for the computation of \( L^2 \) from actual data. The entire vertical region was divided into equal intervals of 500 meters each and the data for missing levels were interpolated linearly. \( L^2 \) values were computed with central difference for intermediate levels and forward and backward difference for the bottom and top levels respectively. Evaluation of \( L_{\alpha p}, L_{\alpha} \) and \( \alpha \) values were done by using the Marquardt method of least square approximation. This is an extension of Gauss-Newton method to allow for the convergence with relatively poor starting guesses for unknown coefficients \( L_{\alpha p}, L_{\alpha} \) and \( \alpha \). In general, the steepest descent procedure also converges for poor starting values, but requires a longer solution time in comparison to the Marquard method. An example of \( L^2 \) profile over Peshawar is shown in Fig.2. Table 1 enlists the dates on which lee-waves were observed over Kashmir valley. Last two columns show the observed and the computed wavelengths.
4. Discussion

The solution for the vertical velocity and displacement, without any restriction on the value of \( k \), is physically significant. Small wavenumber \( 0 < k \leq L_\infty \) give rise to long waves commonly known as hydrostatic mountain waves (\( \geq 30 \text{ km} \)). (Philips and Brown 1983), Doos (1961), while attempting to explain the behaviour of the solution in the domain \( 0 < k < L_\infty \) neglected the contribution to the Fourier integral for values of \( k \) in this interval since he considered their contribution to be very small.

We note from Eqns. (16a & b) that the contribution is proportional to the term \( e^{-kx} \). For small values of \( k \) and \( x \), the value of the integral cannot be considered small enough to be neglected. It will decay exponentially with horizontal distance \( x \) from the mountain top. Vertical propagation of these waves, far deeper into the atmosphere, even up to 25-30 km in the stratosphere, has been observed since long, as lenticular and nacreous (mother of pearl) clouds (Alaka 1958, Queney et al. 1960). Energy propagation to these heights is well explained by qualitatively applying the radiation condition as shown in the Appendix I. Behaviour of these long waves in lower levels have been analysed by various workers (Klemp and Lilly 1975, Smith 1977), AVHRR pictures selected for the validation purposes. Over Jammu & Kashmir region, in the present study, indicate only lee-waves. An experimental case study, however, for observed hydrostatic mountain wave may justify the above view on small wavenumbers.

Lower boundary in Eqn.(10) provides a simple mathematical tool for formulating a function for rugged hilly terrain as in Jammu & Kashmir region. Marquardt's method of least square approximation not only provides the values of constants for such functions, but also provides us the best fit values of \( L_\infty^2, L_0^2 \) and \( \alpha \) for \( L^2 \) profile computed from the observed data. This method of approximation is also useful for computing the resonant wavelength even in the case when low level winds are negative within the shallow layers from the ground as on 14 August 1985.

Appendix I

Application of radiation condition

The general solution of Eqn.(7) for various values of \( k \) is

\[ w = A J_{-ip}(\mu) + BY_{-ip}(\mu) \quad \text{for} \quad k \geq L_\infty \quad (AI.1) \]

\[ w = C J_{-ip}(\mu) + D J_{ip}(\mu) \quad \text{for} \quad k < L_\infty \quad (AI.2) \]
If \( k \geq L_\infty \), \( ip \) is real and in that case as \( z \to \infty \) or \( \mu \to 0 \), \( Y_{-ip} (\mu) \to \infty \). Hence we must drop the second term in (A1.2) to fulfill the radiation condition. When \( k < L_\infty \) solution (A1.2) is given in terms of Bessel functions with imaginary order. Doos (1961), using Boole’s (1844) expansion, erroneously applied the rigid bounding condition and found that the solution is identically zero for \( k < L_\infty \). He finally neglected the contribution to the Fourier integral for values of \( k \) in the interval \( 0 < k < L_\infty \). Behaviour of Bessel function with imaginary argument, however, is very little discussed in the literature. Therefore the following approach is adopted to determine the behaviour of the function when the argument tends to zero:

\[
\text{Limit } J_{ip} (\mu) = \text{Limit}_{\mu \to 0}\left( \frac{1}{ip+1} \right) e^{ip \log (\mu/2)} = \text{Limit}_{\mu \to 0} \frac{1}{ip+1} e^{ip \log (\mu/2)} = \text{Limit} (\text{Constant}) e^{ip \log (\mu/2)} \quad (A1.3)
\]

As \( \mu \to 0 \), \( \log (\mu/2) \to -\infty \), \( p \log (\mu/2) < 0 \) as \( \mu \to 0 \). This indicates that \( J_{ip} (\mu) \) represents a downward travelling wave. Similarly we can show that \( J_{ip} (\mu) \) represents an upward travelling wave. Hence in (A1.2) the second term is to be dropped. \( i.e., D = 0 \) to satisfy the radiation condition. Thus, for both the cases (when \( k \geq L_\infty \) and \( k < L_\infty \)), the uniform solution is

\[ w = A J_{ip} (\mu) \quad (A1.4) \]

where \( A \) is an arbitrary function of \( k \).

References


