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A scheme for deriving grid point values from observatory reports for numerical weather prediction

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ABSTRACT. A mathematical method is developed to calculate the value of any parameter at a grid-point from its average values over three squares symmetric around the point. The weighting functions are derived from the dimensions of the scanning squares. This procedure takes into account all differentials below the sixth order in a Taylor expansion. Truncation error due to neglect of higher order differentials is assessed. Certain refinements to the weighting functions due to using finite number of observations in scanning squares are introduced. Modification and suppression of small waves are discussed in relation to scan-length, average distance between observatories and their random distribution. The method was tried on long and short waves by a numerical experiment.

1. Introduction to objective analysis

Weather data are available for irregularly spaced network of observing stations. Analysis of any weather parameter on the chart is made by drawing a set of isopleths at chosen interval of the parameter values. The run of the isopleths is fixed by visual inspection and interpolation and the isopleths are then smoothed. Prior knowledge of the disposition of isopleths associated with known synoptic scales is also used. The above processes are subjective and to that extent the analysis is bound to be subjective also. The process of smoothing suppresses meso-scale features and allows a study of macro-scale synoptic features. The variation of the meteorological parameters is generally nonlinear and therefore interpolation by inspection as described above may not be very accurate and certainly not unique. Further, for numerical methods of prognosis interpolated values of the parameters are required at fixed grid points. The grid point values of the parameter assessed by interpolation from the above isopleth analysis is likely to be more subjective and uncertain.

Workers in the field of numerical weather prediction have been striving to work out an objective method of analysing the weather chart. By objective method is meant using a specified scheme of interpolation to obtain uniquely the values of the meteorological parameters over a rectangular matrix of locations from an irregularly spaced network of data.

There are basically two distinct and broad approaches to the problem of objective analysis — (*a*) Surface fitting techniques and (*b*) Weighted average correction techniques.

1.1. *Surface fitting techniques*

(i) *Exact fitting* — Panofsky (1949) used the exact surface fitting method for the analysis of synoptic meteorological features. In this method a polynomial of n^{th} degree in x and y,

$$P = \Sigma \, a_{ij} \, x^i y^j \qquad (i + j \leqslant n)$$

which contains $\frac{1}{2}(n+1)(n+2)$ constants is fitted to a field of scalar variable at $\frac{1}{2}(n+1)(n+2)$ observation points and the values of the parameter at fixed grid points in the field are evaluated from the above polynomial. Objective analyses are carried out in France by the approximation of the field throughout the area of the analysis with the help of spherical functions of geographical coordinates.

(ii) *Least square fitting* — As a variation of the above method, a polynomial can also be fitted approximately by the least square fitting technique. Usually a quadratic surface is fitted over a region having data from at least 12 observation points (Cressman 1957). For obtaining a better fit of the analysed grid point value to the values at the stations nearby, squared differences are weighted with a factor inversely proportional to the distance. The weighting factor used is l'/l, where, l

is the distance of the observation point from the grid point and l' that of the nearest observation. Therefore instead of minimising ΣE^2, $\Sigma (l'/l) E^2$ is minimised. E is the difference between the observed and calculated values.

The methods of surface fitting are cumbersome and time consuming and are not generally favoured for routine use.

1.2. *Weighted average correction techniques*

Bergthorssen and Döös (1955) have evolved a method of weighted average correction technique, which has been modified as a simplified iterative scheme by Cressman (1957), and forms the basis of the method used by many for the routine operational numerical analysis. Cressman (1959) later modified the above method as an iterative scheme.

1.3. *An optimal interpolation method*

An optimal interpolation method has been developed by Gandin (1960, 1963) and the USSR School. In this method interpolation weights are determined from data on the auto-correlation function of the parameter being analysed, so that the mean square error of the analysis is minimal.

Though the above method of finding the grid point values is one of weighted average of neighbouring observations, ease of application and rationale of the weighting factors is not satisfactory, except that nearer observations are given a higher weightage. An appreciation of the accuracies of the various techniques is given by Gandin and Lugina (1969). In the following pages the authors have attempted to develop a mathematical model for assessing the grid point values of meteorological parameters.

2. Grid point values from average over an area

The problem is to find the grid point value from those at a number of points around. Let $f(x, y)$ represent the value of the parameter under consideration at any point x and y in the neighbourhood of a grid point, coordinates assumed to be $x = 0$ and $y = 0$. Expanding $f(x, y)$ in the neighbourhood of the grid point,

$$f(x, y) = f + \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}\right) f + \frac{1}{2!}\left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}\right)^2 f + \frac{1}{3!}\left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}\right)^3 f + + \frac{1}{4!}\left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}\right)^4 f + \ldots\ldots \quad (1)$$

where, f, $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$, $\dfrac{\partial^2 f}{\partial x^2}$ etc are the values of the function and its derivatives at $x = 0$, $y = 0$.

The average value of $f(x, y)$ over a square symmetrical around the grid point and of side $2d$ is —

$$\frac{\int_{-d}^{+d}\int_{-d}^{+d} f(x, y)\, dx\, dy}{\int_{-d}^{+d}\int_{-d}^{+d} dx\, dy} = \overline{f_d}$$

Working out a similar mean value of the functions in the right hand side,

$$\begin{aligned}\overline{f_d} &= f + \frac{d^2}{3!}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) + \frac{d^4}{5!}\left(\frac{\partial^4 f}{\partial x^4} + \frac{\partial^4 f}{\partial y^4} + \frac{10}{3}\frac{\partial^4 f}{\partial x^2 \partial y^2}\right) + \ldots \\ &= f + d^2 P + d^4 Q + \ldots\ldots \quad (2)\end{aligned}$$

where, $P = \dfrac{1}{3!}\left(\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2}\right)$ and $Q = \dfrac{1}{5!}\left(\dfrac{\partial^4 f}{\partial x^4} + \dfrac{\partial^4 f}{\partial y^4} + \dfrac{10}{3}\dfrac{\partial^4 f}{\partial x^2 \partial y^2}\right)$

If averaging is done over three squares of sides $2d$, $2e$ and $2g$ respectively, neglecting the sixth and higher order differentials we get,

$$\bar{f}_d = f + d^2 P + d^4 Q, \quad \bar{f}_e = f + e^2 P + e^4 Q, \quad \bar{f}_g = f + g^2 P + g^4 Q \tag{3}$$

The set of Eqns. (3) can be solved to give a unique value for $\bar{f}$ in terms of $\bar{f}_d$, $\bar{f}_e$ and $\bar{f}_g$.

$$f = \omega_d \bar{f}_d + \omega_e \bar{f}_e + \omega_g \bar{f}_g \tag{4}$$

$$\omega_d = \frac{e^2 g^2 (g^2 - e^2)}{B}, \quad \omega_e = \frac{g^2 d^2 (d^2 - g^2)}{B}, \quad \omega_g = \frac{d^2 e^2 (e^2 - d^2)}{B} \tag{5}$$

$$B = (d^2 - e^2)(e^2 - g^2)(g^2 - d^2) \tag{6}$$

This would suggest that the value of the parameter at a grid point can be assessed to an accuracy of fifth order differentials of the function representing the parameter in its neighbourhood by suitably weighting the means of observations around the grid point. In the above scheme the average values of the function within three squares symmetric around the origin, $\bar{f}_d$, $\bar{f}_e$ and $\bar{f}_g$, are found and these are weighted by 'weighting factors' ω_d, ω_e and ω_g (which are functions of d, e and g only) respectively to find the values of the parameter at the grid point. In practice the values of $\bar{f}_d$ etc have to be taken as the mean of the values at the available observatories within the appropriate squares. The difference between $\bar{f}_d$ and the average of finite number of points will be dealt with in a later section. At this stage it may be mentioned that the above method assumes Taylor expansion of the function to be valid in the neighbourhood, and that the sum of terms beyond the fifth is comparatively negligible. It will however be clear that in the case of a wave function the accuracy is dependent on the wave length with reference to the scan distance.

'Scans' within a circular field around the grid point also give similar results, but d, e and g now refer to the radii of the scan circles. But in terms of computer time, it is preferable to use a square field.

It is interesting to note that the above method of assessing $f(0, 0)$ as a weighted mean of three scan averages is equivalent to fitting a fifth order polynomial for the field in the neighbourhood and assessing $f(0, 0)$ from this polynomial.

3. Truncation error

In the above method of calculating $f(0, 0)$, there will be a truncation error due to neglect of differential higher than the fourth. To illustrate this point we shall expand the right hand side of $\bar{f}_d$, $\bar{f}_e$ and f_g (*vide* Eq. 2) fully as—

$$\begin{aligned}\bar{f}_d = f &+ \frac{d^2}{3!}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) + \frac{d^4}{4!}\left(\frac{1}{5}\frac{\partial^4 f}{\partial x^4} + \frac{{}^4C_2}{3^2}\frac{\partial^4 f}{\partial x^2 \partial y^2} + \frac{1}{5}\frac{\partial^4 f}{\partial y^4}\right) + \\ &+ \frac{d^6}{6!}\left(\frac{1}{7}\frac{\partial^6 f}{\partial x^6} + \frac{{}^6C_2}{3\times5}\frac{\partial^6 f}{\partial x^4 \partial y^2} + \frac{{}^6C_2}{3\times5}\frac{\partial^6 f}{\partial x^2 \partial y^4} + \frac{1}{7}\frac{\partial^6 f}{\partial y^6}\right) + \frac{d^8}{8!}\left(\frac{1}{9}\frac{\partial^8 f}{\partial x^8} + \right. \\ &\left. + \frac{{}^8C_2}{3\times7}\frac{\partial^8 f}{\partial x^6 \partial y^2} + \frac{{}^8C_4}{5\times5}\frac{\partial^8 f}{\partial x^4 \partial y^4} + \frac{{}^8C_6}{3\times7}\frac{\partial^8 f}{\partial x^2 \partial y^6} + \frac{1}{9}\frac{\partial^8 f}{\partial y^8}\right) + \ldots\ldots\end{aligned} \tag{7}$$

and two similar terms for $\bar{f}_e$ and $\bar{f}_g$.

The error E in the value of f calculated by (4) will be —

$$-E = \frac{1}{6!}\left(\omega_d d^6 + \omega_e e^6 + \omega_g g^6\right)\left(\frac{1}{7}\frac{\partial^6 f}{\partial x^6} + \frac{{}^6C_2}{3\times5}\frac{\partial^6 f}{\partial x^4 \partial y^2} + \frac{{}^6C_4}{3\times5}\frac{\partial^6 f}{\partial x^2 \partial y^4} + \right.$$

$$+ \frac{1}{7} \frac{\partial^6 f}{\partial y^6}\Big) + \frac{1}{8!}\left(\omega_d\, d^8 + \omega_e\, e^8 + \omega_g\, g^8\right)\left(\frac{1}{9}\frac{\partial^8 f}{\partial x^8} + \frac{{}^8C_2}{3\times7}\frac{\partial^8 f}{\partial x^6 \partial y^2} + \right.$$

$$\left. + \frac{{}^8C_4}{5\times5}\frac{\partial^8 f}{\partial x^4 \partial y^4} + \frac{{}^8C_6}{3\times7}\frac{\partial^8 f}{\partial x^2 \partial y^6} + \frac{1}{9}\frac{\partial^8 f}{\partial y^8}\right) + \ldots\ldots \tag{8}$$

In order to get a rough estimate of the order and nature of the truncation error, let us assume a simple sine function to represent the spatial distribution of the parameter. We take,

$$f(x, y) = \sin\left(\frac{2\pi}{\lambda}x + \phi\right) \tag{9}$$

for all y and substitute in (8). The value of E can then be compared with $f(0,0) = f = \sin\phi$.

$$\frac{E}{\sin\phi^*} = \left(\frac{2\pi}{\lambda}\right)^6 \frac{1}{7!}\left(\omega_d\, d^6 + \omega_e\, e^6 + \omega_g\, g^6\right) - \left(\frac{2\pi}{\lambda}\right)^8 \frac{1}{9!}\left(\omega_d\, d^8 + \omega_e\, e^8 + \omega_g\, g^8\right) + \ldots \tag{10}$$

Setting $\frac{2\pi d}{\lambda} = \alpha$, $\frac{2\pi e}{\lambda} = \beta$ and $\frac{2\pi g}{\lambda} = \gamma$

$$\frac{E}{\sin\phi} = \frac{\omega_d}{\alpha}\left(\frac{\alpha^7}{7!} - \frac{\alpha^9}{9!} + \frac{\alpha^{11}}{11!} - \ldots\ldots\ldots\right) +$$

$$+ \frac{\omega_e}{\beta}\left(\frac{\beta^7}{7!} - \frac{\beta^9}{9!} + \frac{\beta^{11}}{11!} - \ldots\ldots\ldots\right) +$$

$$+ \frac{\omega_g}{\gamma}\left(\frac{\gamma^7}{7!} - \frac{\gamma^9}{9!} + \frac{\gamma^{11}}{11!} - \ldots\ldots\ldots\right)$$

$$= \omega_d\left(-\frac{\sin\alpha}{\alpha} + 1 - \frac{\alpha^2}{3!} + \frac{\alpha^4}{5!}\right) + \omega_e\left(-\frac{\sin\beta}{\beta} + 1 - \frac{\beta^2}{3!} + \frac{\beta^4}{5!}\right) +$$

$$+ \omega_g\left(-\frac{\sin\gamma}{\gamma} + 1 - \frac{\gamma^2}{3!} + \frac{\gamma^4}{5!}\right) \tag{11}$$

The ratio of the dimensions of the 'scanning' squares to the wavelength of the function determines the truncation errors. Table 1 gives the values of $E/\sin\phi$ for certain values of g/λ and $d:e:g$.

TABLE 1

Wave of $E/\sin\phi$ (Truncation error)

g/λ	$g/e = 2,\ e/d = 2$	$g/e = 3,\ e/d = 3$
1	+0·08	—0·016
0·5	+0·004	+0·0003

Either by making the largest dimensions of the scanning square half the wavelength or by taking a greater ratio between the scanning squares, the error can be kept below two per cent. For short wavelengths, the truncation error increases very rapidly as the fourth power of the ratios of scan lengths to the wavelength.

*In the special case of sin $\phi=0$ or being some integral multiple of E can be discussed by transferring sin ϕ to the right hand side.

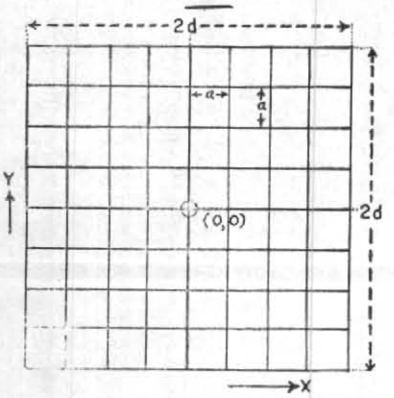

Fig.1

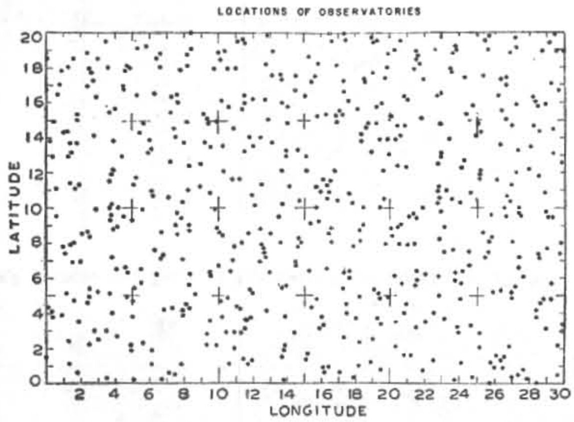

Fig. 2

4. Difference in mean due to the finite number of points in scanning squares

To take into account the positions of observatories in a scanning square, we may express their coordinates as the sum of a regular spacing and a random variation. Consider a lattice with separation a between successive lines. The lines are also numbered starting from 0 through the origin, positive on increasing sides of x and y and negative on decreasing sides. If the observatories were regularly spaced they would be located at the intersections of the lines. The observatory which should be at the intersection of the r^{th} ($x =$ constant) and s^{th} ($y =$ constant) lines be regarded as displaced by $\epsilon_{r(s)}$ in the x-direction and $\epsilon_{s(r)}$ in the y-direction. All the ϵ's are random numbers with zero mean and σ_ϵ standard deviation. Thus the coordinates of the (r, s) observatory will be —

$$x(r, s) = ar + \epsilon_{r(s)} \qquad\qquad y(r, s) = as + \epsilon_{s(r)}$$

$$\begin{aligned} f\left\{x(r, s), y(r, s)\right\} = {} & f + \left(ar + \epsilon_{r(s)}\right)\frac{\partial f}{\partial x} + \left(as + \epsilon_{s(r)}\right)\frac{\partial f}{\partial y} + \frac{1}{2!}\left(ar + \epsilon_{r(s)}\right)^2\frac{\partial^2 f}{\partial x^2} + \\ & + \frac{1}{2!}\left(as + \epsilon_{s(r)}\right)^2\frac{\partial^2 f}{\partial y^2} + \frac{1}{2!}\left(ar + \epsilon_{r(s)}\right)\left(as + \epsilon_{s(r)}\right)\frac{\partial^2 f}{\partial x\,\partial y} + \\ & + \frac{1}{3!}\left(ar + \epsilon_{r(s)}\right)^3\frac{\partial^3 f}{\partial x^3} + \frac{{}^3C_1}{3!}\left(ar + \epsilon_{r(s)}\right)^2\left(as + \epsilon_{s(r)}\right)\frac{\partial^3 f}{\partial x^2\,\partial y} + \frac{{}^3C_2}{3!}\left(ar + \epsilon_{r(s)}\right)\times \\ & \times\left(as + \epsilon_{s(r)}\right)^2\frac{\partial^3 f}{\partial x\,\partial y^2} + \frac{1}{3!}\left(as + \epsilon_{s(r)}\right)^3\frac{\partial^3 f}{\partial y^3} + \frac{1}{4!}\left(ar + \epsilon_{r(s)}\right)^4\frac{\partial^4 f}{\partial x^4} + \\ & + \frac{{}^4C_1}{4!}\left(ar + \epsilon_{r(s)}\right)^3\left(as + \epsilon_{s(r)}\right)\frac{\partial^4 f}{\partial x^3\,\partial y} + \frac{{}^4C_2}{4!}\left(ar + \epsilon_{r(s)}\right)^2\left(as + \epsilon_{s(r)}\right)^2\frac{\partial^4 f}{\partial x^2\,\partial y^2} + \\ & + \frac{{}^4C_3}{4!}\left(ar + \epsilon_{r(s)}\right)\left(as + \epsilon_{s(r)}\right)^3\frac{\partial^4 f}{\partial x\,\partial y^3} + \frac{1}{4!}\left(as + \epsilon_{s(r)}\right)^4\frac{\partial^4 f}{\partial y^4} + \ldots\ldots \quad (12) \end{aligned}$$

Each of the terms on the right hand side has to be averaged between $r = -n$ to $+n$ and $s = -n$ to $+n$. It may be noted that $na = d$.

As $r, s, \epsilon_{r(s)}$ and $\epsilon_{s(r)}$ are mutually uncorrelated, it can be shown that for integral values of N, product moment of any two of them each raised to odd or even powers is as given below;

$$\overline{r^{2N}\,\epsilon_{r(s)}^{2N+1}} = 0, \qquad \overline{r^{2N+1}\,\epsilon_{r(s)}^{2N}} = 0, \qquad \overline{r^{2N+1}\,\epsilon_{r(s)}^{2N+1}} = 0$$

$$\overline{r^{2N}\,\epsilon_{r(s)}^{2N}} = \overline{r^{2N}}\;\overline{\epsilon_{r(s)}^{2N}}, \qquad \overline{r^{2N}\,s^{2N+1}} = \overline{r^{2N+1}}\;\overline{s^{2N}} = 0, \qquad \overline{r^{2N}\,s^{2N}} = \overline{r^{2N}}\;\overline{s^{2N}}$$

Hence,

$$\begin{aligned}\overline{f\Big[\,x\,(r,s),\; y\,(r,s)\,\Big]} = {} & f + \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)\left\{\frac{d^2}{6}\left(1+\frac{1}{n}\right) + \frac{1}{2}\,\sigma_\epsilon^2\right\} + \left(\frac{\partial^4 f}{\partial x^4} + \frac{\partial^4 f}{\partial y^4}\right) \\ & \times \left\{\frac{d^4}{4!}\,\frac{(n+1)\,(3n^2+3n-1)}{15\,n^3} + \frac{{}^4C_2}{4!}\,\frac{d^2}{n^2}\,\frac{1}{3}\,n(n+1)\sigma_\epsilon^2 + \frac{3}{4!}\,\sigma_\epsilon^4\right\} \\ & + \frac{\partial^4 f}{\partial x^2\,\partial y^2}\left\{\frac{{}^4C_2}{4!}\,\frac{d^4}{9}\,\frac{(n+1)^2}{n^2} + \frac{{}^4C_2}{4!}\,\frac{d^2}{n^2}\,\frac{2}{3}\,n\,(n+1)\,\sigma_\epsilon^2 + \frac{1}{4!}\,\sigma_\epsilon^4\right\} \\ & + \;\ldots\ldots\ldots \end{aligned} \tag{13}$$

Putting, $\sigma_\epsilon = \dfrac{d}{n}\,X$

$$\begin{aligned}\overline{f\,\Big[x\,(r,s),\, y\,(r,s)\Big]} = {} & f + \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)\frac{d^2}{2}\left\{\frac{1}{3}\left(1+\frac{1}{n}\right) + \frac{X^2}{n^2}\right\} + \\ & + \left(\frac{\partial^4 f}{\partial x^4} + \frac{\partial^4 f}{\partial y^4}\right)\frac{d^4}{4!}\left\{\frac{1}{15}\left(3+\frac{6}{n}+\frac{2}{n^2}-\frac{1}{n^3}\right) + \right. \\ & \left. + \frac{{}^4C_2}{3}\left(\frac{1}{n^2}+\frac{1}{n^3}\right)X^2 + 3\,\frac{X^4}{n^4}\right\} + \left(\frac{\partial^4 f}{\partial x^2\,\partial y^2}\right)\frac{d^4}{4!}\left\{\frac{{}^4C_2}{9}\left(1+\right.\right. \\ & \left.\left. + \frac{2}{n} + \frac{1}{n^2}\right) + \frac{2}{3}\,{}^4C_2\left(\frac{1}{n^2}+\frac{1}{n^3}\right)X^2 + \frac{X^4}{n^4}\right\} + \ldots\ldots \end{aligned} \tag{14}$$

X represents σ_ϵ as a function of the mean distance between stations. If we assume $X \leqslant \frac{1}{2}$ and justifiably neglect terms involving n^{-2}, n^{-3} and n^{-4}

$$\overline{f\,\Big[\,x\,(r,s),\, y\,(r,s)\Big]} = f + \left(1+\frac{1}{n}\right)d^2\,P + \left(1+\frac{2}{n}\right)d^4\,Q \tag{15}$$

This equation differs from Eq.(2) in that the coefficients of P and Q (*i.e.*, d^2 and d^4 of Eq. 2) are now multiplied by $(1+1/n)$ and $(1+2/n)$ respectively. The three equations of Eq. (3) as modified by the factors $(1+1/n)$ and $(1+2/n)$ now form the appropriate equations to solve for f. Depending upon the value of n which is related to the number of observatories in the scanning squares, equations (14) and (15) may be used.

5. Modification and suppression of waves

Let us consider the effect of the method set out earlier, *viz.*, the method of weighted averaging, on modifying the actual values of the parameter at the grid points. For simplicity, let us consider a one-dimensional field in which the parameter is distributed according to a sine-formula, say, $f = \sin\,[(2\pi/\lambda)\,x + \phi]$. The assessed mean grid point value at $X = 0$ is :

$$\int_{-d}^{+d} \sin\left(\frac{2\,\pi}{\lambda}\,x + \phi\right)dx \Big/ \int_{-d}^{+d} dx = \sin\phi\;\frac{\sin\dfrac{2\,\pi\,d}{\lambda}}{\dfrac{2\,\pi\,d}{\lambda}} \tag{16}$$

Table 2 shows the value of $\sin\left(\frac{2\pi d}{\lambda}\right)\Big/\left(\frac{2\pi d}{\lambda}\right)$ for different values of d/λ from 0·1 to 1·5. The value of $\bar{f}_d$ is zero when $d/\lambda = N/2$, where N is any integer. As e and g are usually integral multiples of d, the values of $\bar{f}_e$ and $\bar{f}_g$ will also be zero. Cases of $\bar{f}_d = \bar{f}_e = \bar{f}_g = 0$ in the system of equations (3) leads to the solution of f=0. This is equivalent to such waves being suppressed. Hence all waves of wavelength λ_s=$2d/N$ where d is the lowest scan length, will be suppressed.

The cases of other short wavelengths can also be considered. For a given ratio of $d:e:g$, the weighting factors ω_d, ω_e and ω_g are fixed. Hence once the ratio d/λ is fixed, the value derived by Eq. (3) for the grid point may be written as —

$$f_c = \sin\phi\left(\omega_d\frac{\sin\frac{2\pi d}{\lambda}}{\frac{2\pi d}{\lambda}} + \omega_e\frac{\sin\frac{2\pi e}{\lambda}}{\frac{2\pi e}{\lambda}} + \omega_g\frac{\sin\frac{2\pi g}{\lambda}}{\frac{2\pi g}{\lambda}}\right)$$

As f is actually $\sin\phi$ at the grid-point,

$$\frac{f_c}{f} = \omega_d\frac{\sin\frac{2\pi d}{\lambda}}{\frac{2\pi d}{\lambda}} + \omega_e\frac{\sin\frac{2\pi e}{\lambda}}{\frac{2\pi e}{\lambda}} + \omega_g\frac{\sin\frac{2\pi g}{\lambda}}{\frac{2\pi g}{\lambda}} \tag{17}$$

where, f_c is the calculated and f the actual value of the parameter at the grid point.

For the ratio $d/e = e/g = 1/2$, $f_c\,/\,f$ is also tabulated in Table 2. The chief points to note are that when the dimension of the shortest scan becomes comparable with or less than the wavelength of the sine wave, the computed value at the grid-point is only a small fraction of the actual value, that is a partial or a more complete suppression of the wave occurs. In some cases, the sign of the calculated value may be systematically opposite of the actual. Waves, a few times shorter than the shortest scan distance are completely suppressed.

We may now consider modification and suppression of waves by averaging the values at a finite number of observing points. Considering again in one dimension and writing $x = ar + \epsilon_r$,

$$\overline{f_{an}} = \frac{1}{2n+1}\sum_{r=-n}^{r=+n}\sin\left\{\frac{2\pi}{\lambda}\left(ar+\epsilon_r\right)+\phi\right\}$$

$$= \frac{1}{2n+1}\sum_{r=-n}^{r=+n}\left\{\sin\left(\frac{2\pi}{\lambda}\,ar+\phi\right)\cos\frac{2\pi\epsilon_r}{\lambda} + \cos\left(\frac{2\pi}{\lambda}\,ar+\phi\right)\sin\frac{2\pi\epsilon_r}{\lambda}\right\} \tag{18}$$

As discussed earlier $\sin\left(\frac{2\pi}{\lambda}\,ar+\phi\right)$, or $\cos\left(\frac{2\pi}{\lambda}\,ar+\phi\right)$ and $\cos\left(\frac{2\pi\,\epsilon_r}{\lambda}\right)$ or $\sin\left(\frac{2\pi\,\epsilon_r}{\lambda}\right)$ are uncorrelated. Hence,

$$\overline{f_{an}} = \left\{\frac{1}{2n+1}\sum_{r=-n}^{r=+n}\sin\left(\frac{2\pi}{\lambda}\,ar+\phi\right)\right\}\left\{\frac{1}{2n+1}\sum_{r=-n}^{r=+n}\cos\frac{2\pi}{\lambda}\,\epsilon_r\right\} +$$

$$+ \left\{\frac{1}{2n+1}\sum_{r=-n}^{r=+n}\cos\left(\frac{2\pi}{\lambda}\,ar+\phi\right)\right\}\left\{\frac{1}{2n+1}\sum_{r=-n}^{r=+n}\sin\frac{2\pi}{\lambda}\,\epsilon_r\right\}$$

TABLE 2

Ratio of calculated values of parameter to actual value for a sine wave

d/λ	$\frac{\sin(2\pi d/\lambda)}{2\pi d/\lambda}$	f_c/f
·1	0·95	1·00
·2	0·77	0·98
·3	0·51	0·79
.4	0·24	0·42
·5	0·0	0.0
·6	—0·16	—0·29
·7	—0·22	—0·34
·8	—0·19	—0·24
·9	—0·10	—0·10
1·0	0·0	0·0
1·1	+0·09	+0·10
1·2	+0·13	+0·16
1·3	+0·12	+0·19
1·4	+0·07	+0·13
1·5	0·0	0·0

d=The shortest scan length, λ=Wavelength

$e : d = g : e = 2$

TABLE 3

Ratio of the calculated value of the parameter to actual value

a/λ	f_c'/f
1/3	—0·04
1/4	—0·14
1/5	0·05
1/6	0·24
1/7	0·49
1/8	0·77
1/9	0·86
1/10	0·90
1/11	0·94
1/12	0·98
1/13	0·99
1/14	0·99
1/15	1·00

a = Grid length=Mean spacing of observatories

Shortest scan length=$2a=d$ $\sigma_\epsilon = a$

λ=Wavelength $d : e : g = 1 : 2 : 4$

If ϵ_r is normally distributed, it can be shown that for an infinite sample,

$$\overline{\cos \frac{2\pi}{\lambda}\epsilon_r} = \exp\left(-\frac{2\pi^2\sigma_\epsilon^2}{\lambda^2}\right) \quad \text{and} \quad \overline{\sin\frac{2\pi}{\lambda}\epsilon_r} = 0$$

We write,

$$\frac{1}{2n+1}\sum_{r=-n}^{r=+n}\cos\left(\frac{2\pi\epsilon_r}{\lambda}\right) \simeq \exp\left(-\frac{2\pi^2\sigma_\epsilon^2}{\lambda^2}\right)$$

$$\frac{1}{2n+1}\sum_{r=-n}^{r=+n}\sin\left(\frac{2\pi\epsilon_r}{\lambda}\right) \simeq 0.$$

Hence,

$$\overline{f_{a_n}} = \left[\exp\left(\frac{2\pi^2}{\lambda^2}\sigma_\epsilon^2\right)\right]\left(\sin\phi\right)\frac{1}{2n+1}\cdot\left\{2\cos\frac{n\pi a}{\lambda}\sin\frac{(n+1)\pi a}{\lambda}\operatorname{cosec}\frac{\pi a}{\lambda} - 1\right\} \quad (19)$$

provided a/λ is not an integer.

$\sin\phi$, $\cos\dfrac{n\pi a}{\lambda}$ and $\sin\dfrac{(n+1)\pi a}{\lambda}$ cannot exceed 1. Except in a very narrow range of value of a/λ (the neighbourhood of integral values of a/λ which is excluded) $\operatorname{cosec}\pi a/\lambda$ cannot counteract the effect of $\exp\left(-\dfrac{2\pi^2\sigma_\epsilon^2}{\lambda^2}\right)$. When λ is equal to σ_ϵ or less, $\bar{f}_{an} \simeq 0$. We may define a λ_c as $\exp\left(-\dfrac{2\pi^2\sigma_\epsilon^2}{\lambda_c^2}\right) = 0\cdot01$ and regard all values of $\lambda \leqslant \lambda_c$ as suppressed. This gives,

$$\lambda_c/\sigma_\epsilon = \sqrt{\frac{-2\pi^2}{\ln 0.01}} = \frac{\pi}{1.5} \simeq 2 \tag{20}$$

All wavelengths equal to twice σ_ϵ or shorter are suppressed on account of the random distribution of the observatories. Eq. (20) is not valid when a/λ is an integer. But a similar result can be derived for these cases also. It appears that random distribution of observation points is a powerful tool to suppress small waves. In the case of continuous integration, suppression of waves was dependent upon the relationship of scan distances to wave length. And discrete randomised observatories leads to suppression of small waves when the wavelength is less than $2\sigma_\epsilon$. The condition of $2\,an/\lambda = 2d/\lambda$ being integer does not now lead to suppression of such waves.

The retrieval of wave form for different values of λ can also be studied in this case.

$$\bar{f}_{d=an} = \left[\exp\left(-\frac{2\pi^2\sigma_\epsilon^2}{\lambda^2}\right)\right](\sin\phi)\frac{1}{2n+1}\left\{2\cos\frac{n\pi a}{\lambda}\sin\frac{(n+1)\pi a}{\lambda}\operatorname{cosec}\frac{\pi a}{\lambda} - 1\right\}$$

$$\bar{f}_{e=2an} = \left[\exp\left(-\frac{2\pi^2\sigma_\epsilon^2}{\lambda^2}\right)\right](\sin\phi)\frac{1}{4n+1}\left\{2\cos\frac{2n\pi a}{\lambda}\sin\frac{(2n+1)\pi a}{\lambda}\operatorname{cosec}\frac{\pi a}{\lambda} - 1\right\}$$

$$\bar{f}_{g=4an} = \left[\exp\left(-\frac{\pi^2\sigma_\epsilon^2}{\lambda^2}\right)\right](\sin\phi)\frac{1}{8n+1}\left\{2\cos\frac{4n\pi a}{\lambda}\sin\frac{(4n+1)\pi a}{\lambda}\operatorname{cosec}\frac{\pi a}{\lambda} - 1\right\} \tag{21}$$

Representing the calculated grid-point value as f_c',

$$f_c' = \omega_d f_d + \omega_e f_e + \omega_g f_g \,.$$

For different ratios of a/λ, values of f_c'/f are tabulated in Table 3, taking $\sigma_\epsilon = a$

As the wavelength increases in proportion to the grid-length (equal to σ_ϵ) the ratio of f_c to the actual value increases. Only when the wavelength is eight times the grid-distance, the calculated amplitude will be about three quarters of the actual amplitude. When a single wave is present, the calculated amplitude of the wave is no doubt less than its original value but as their ratio is the same at all points the shape of the wave is not distorted. When two waves of different wavelengths are present, their amplitudes are modified to differeent extends so that the shape of the resulting wave based on calculated values is distorted from the original.

6. Numerical experiments

The above scheme of finding grid-point values was tried in a numerical experiment. A x, y grid of 31×21 at equal spacing was used. On the average one observatory was alloted to each grid square. The positions of the observatories were randomized by fixing them at distances of ϵ_r along the X-axis and of ϵ_s along the Y-axis from the chosen grid point. ϵ_r and ϵ_s are random numbers with zero mean and standard deviation (σ_ϵ) equal to one grid length. Fig. 2 gives the actual distribution of observatories in the chosen field. Values of the parameter at the observatories were derived according to an assumed function. From these observatory values, the values at the grid-points were calculated by the above scheme and compared with the values derived from the assumed distribution function.

The smallest scan distance was taken as two grid-lengths ($d=2a$) and $e/d = g/e = 2$. With this scheme and using Eq. (14), the weighting factors ω_d, ω_e and ω_g are respectively $1\cdot93$, $-1\cdot00$ and $0\cdot07$. There is

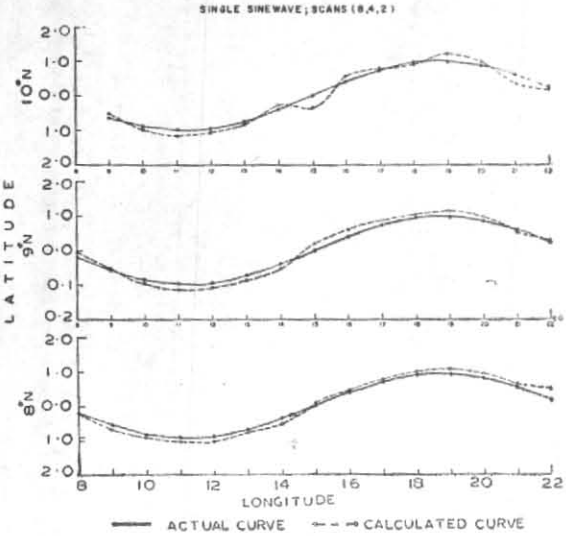

Fig. 3

Actual and computed values of single sine wave using scan lengths 2, 4 and 8

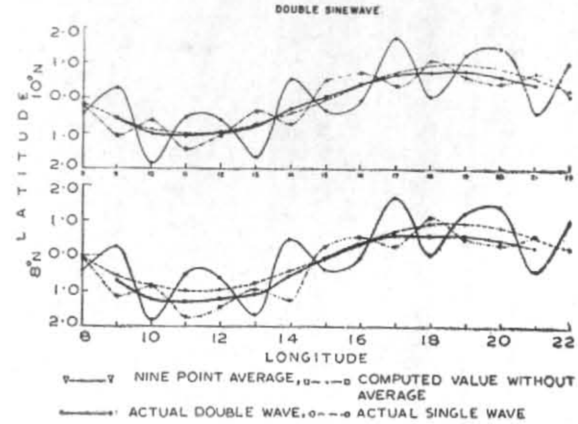

Fig. 5

Comparison of nine point averages of computed waves in the case of double sine wave

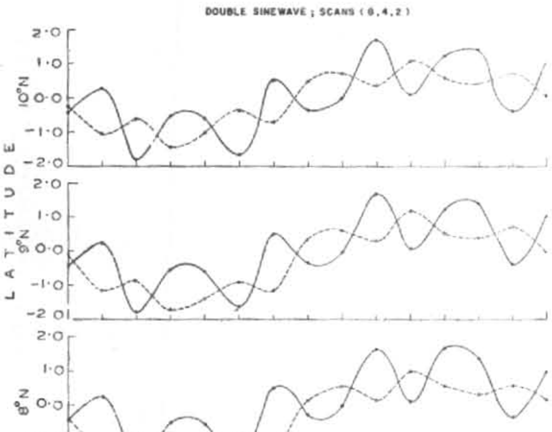

Fig. 4

Actual and computed values of double sine wave using scan lengths 2, 4 and 8

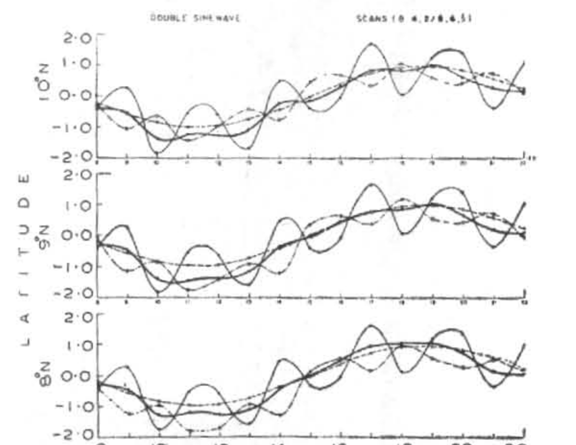

Fig. 6

Comparison of double sine wave for scan lengths of 2, 4 and 8 with 3, 6 and 8 a slight approximation in arriving at these weighting factors. In equation (14), the coefficient of $\partial^4 f/\partial x^2 \partial y^2$ is not the same multiple of the coefficient of $(\partial^4 f/\partial x^4) + (\partial^4 f/\partial y^4)$ in the three scans. But as the differences are not very large and as the influence of the largest scans is very small the error on account of this approximation is negligible. As the outermost scan was over a square of side 16 grid-lengths, grid-point values could be calculated only for less number of points than the grid-points.

The parametric distribution functions tried were —

$$(i)\ f(x, y) = \left[1 - 0{\cdot}01\,(y - 10)^2\right] \sin \frac{2\pi x}{15} \tag{22}$$

$$\text{and}\ (ii)\ f(x, y) = \left[1 - 0{\cdot}01\,(y - 10)^2\right] \left(\sin \frac{2\pi x}{15} + \sin \frac{2\pi x}{2{\cdot}7}\right) \tag{23}$$

The distance x was measured in units of grid-length.

By the method of three scans and weighted average, the values at the grid-points were calculated and compared with the grid-point values expected from the known functional formula. This was done both for a single sine wave (Eq. 22) and for a sinewave with a superposed short wave (Eq. 23). Scan length sets of 2, 4, 8 as well as 3, 6, 8 were used. A pictorial comparison of the calculated and actual values along the chosen X-axes are given in Figs. 3, 4, 5 and 6.

The relevant features noticed about the calculated values are —

(*i*) The mean deviation in the case of single wave of wavelength 15 grid is about 0·12 of the amplitude.

(*ii*) Computed amplitude of the wave is 10 to 20 per cent more than the actual value.

(*iii*) Abrupt differences in errors are some times noticed between neighbouring points.

(*iv*) When the short and long waves (both of equal amplitude) are introduced, the amplitude of the short wave is reduced considerably (to less than 50 per cent) but the wave is not eliminated completely.

(*v*) When nine-point smoothing is carried out, the short wave is completely suppressed and the calculated values correspond well with the expected values of single long wave. Hence the method of computation combined with nine-point smoothing seems to be quite successful in suppressing waves of short wavelength and representing the long wave pattern truthfully.

The discrepancies between actual and computed values, particularly the partial suppression of short waves, may be due to the rather small number of observations in the smallest scanning square as against the infinite population assumed for the theoretical derivations.

7. Conclusions

1. Preliminary trials indicate that the mathematical method developed in the note could be used to calculate grid-point values to a good degree of accuracy.

2. The method consists of simple averaging of values in three separate scans and hence takes less computational time than the other methods. From the low weighting factors for the largest scan, it would appear that even two scans are sufficient.

3. This method is capable of suppressing short waves.

4. The method is not empirical but based on mathematical formulation.

REFERENCES

Bergthorssen, P. and Döös, B.	1955	*Tellus*, **7**, 3, pp. 329-340.
Cressman, G. P.	1957	'An objective analysis study' Tech. Memo. No. 12, Joint Numerical Weather Prediction Unit.
	1959	*Mon. Weath. Rev.*, **87**, 10, pp. 367-374.
Gandin, L. S.	1960	'The optimum interpolation and extrapolation of meteorological fields,' Glavnaia Geofiz. Obs., Trudy, 114, pp. 75-89.
	1963	The Objective Analysis of Meteorological Fields', Leningrad Gidrometeoizdat.
Gandin, L. S. and Lugina, K. M.	1969	A comparison of the accuracy of objective analysis', *WMO Bull.*, **18**, 2, pp. 86-90.
Panofsky, H. A.	1949	*J. Met.*, **6**, pp. 386-392.